GENERALIZED RICCI CURVATURE BOUNDS FOR THREE DIMENSIONAL CONTACT SUBRIEMANNIAN MANIFOLDS

ANDREI AGRACHEV AND PAUL LEE

Abstract. Measure contraction property is one of the possible generalizations of Ricci curvature bound to more general metric measure spaces. In this paper, we discover sufficient conditions for a three dimensional contact subriemannian manifold to satisfy this property.

1. Introduction

In the past few years, several connections between the optimal transportation problems and curvature of Riemannian manifolds were found. One of them is the use of optimal transportation for an alternative definition of Ricci curvature lower bound developed in a series of papers [23, 9, 26]. Based on the ideas in these papers, a generalization of Ricci curvature lower bound for general metric measure spaces, called curvature dimension condition, is introduced in [16, 17, 24, 25, 20]. However the conditions are not easy to check and there is no new example. Recently the case of a Finsler manifold was studied in [21], but the result is very similar to that of the Riemannian case due to strict convexity of the corresponding Hamiltonian. The situation changes dramatically in the case of a subriemannian manifold. The reason is that the class of metric spaces we are dealing with have Hausdorff dimensions strictly greater than their topological dimensions. Therefore, the interplay of the metrics and the measures for these spaces should be significantly different from that of the Riemannian or Finsler case. One particular case of subriemannian manifolds, the Heisenberg group, is studied in [12]. In this case the the space does not satisfy any curvature dimension condition. However, it satisfies a weaker condition, a modification of the so called measure contraction property (see Section 4 for the definition).

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Measure contraction property is another generalization of Ricci curvature lower bound introduced in [25]. More precisely the following holds (see [25]):

Theorem 1.1. *If M is a Riemannian manifold of dimension n, then M satisfies the measure contraction property $MCP(k,n)$ if and only if the Ricci curvature of M is bounded below by k.*

The approach used by [12] relies on the complete integrability of the subriemannian geodesic flow on the Heisenberg group. Because of this the changes in the measure along the geodesic flow can be written down explicitly in this case, which is not possible for subriemannian manifolds in general.

The goal of this paper is to study a subriemannian version of the measure contraction property for all three dimensional contact subriemannian manifolds, generalizing the corresponding result in [12]. This study uses a subriemannian generalization of the classical Ricci curvature. This generalized curvature was introduced by the first author in the 90s for some special cases (including the three dimensional contact subriemannian structures), and in full generality by the first author and I. Zelenko (see [4]). The nature of this invariant is dynamical rather than metrical: the generalized Ricci curvature is simplest differential invariant for the geodesic flow defined on the cotangent bundle T^*M equipped with the bundle structure $\pi : T^*M \to M$. The generalized Ricci curvature is easy to compute and we study its role in the measure contraction property in this paper.

The structure of this paper is as follows. Section 1 gives several basic notions on subriemannian geometry necessary for the present work. In Section 2 we recall and specialize the recent result of [14, 15] on the curvature type invariants of subriemannian manifolds to the three dimensional contact case. The explicit formulas for these invariants will be written down in Section 3. Section 4 contains the main theorem (Theorem 4.5) which shows that if the generalized curvatures are bounded below by a constant, then the subriemannian manifold satisfies a generalized measure contraction property $MCP(r;2,3)$ (see Definition 4.4 below). In particular, if the generalized curvatures are non-negative, then the subriemannian manifold satisfies the measure contraction property $MCP(0,5)$. As a consequence, these spaces satisfy the doubling property and a local Poincaré inequality. In Section 5 we specialize to the case where the contact subriemannian manifolds are related to the isoperimetric problems or particles in magnetic fields. In this case the subriemannian manifold M is the total space of a principle bundle $\pi_M : M \to N$. The base space N is a smooth

surface equipped with a Riemannian metric descending from the subriemannian metric of the total space M. The total space M satisfies the measure contraction property $MCP(\kappa; 2, 3)$ if the surface N has Gauss curvature bounded below by κ (Theorem 5.1). In particular, this is applicable to the two famous examples: the Heisenberg group and the Hopf fibration.

2. Subriemannian Geometry

In this section several basic notions in subriemannian geometry needed in this paper will be introduced (see [19] for more detail). Recall that a Riemannian manifold is a manifold M together with a fibrewise inner product defined on the tangent bundle TM. The length of a curve is defined by this inner product and the Riemannian distance between two points is the length of the shortest curve connecting them. For a subriemannian manifold the fibrewise inner product is defined on a family of subspaces Δ inside the tangent bundle TM. Therefore, the notion of length can only be defined for curves which are tangent to this family Δ. These curves are called horizontal curves and the subriemannian distance between two points is the length of the shortest horizontal curve connecting them.

More precisely a subriemannian manifold is a triple (M, Δ, g), where M is a smooth manifold, Δ is a distribution (a vector subbundle Δ of the tangent bundle TM of the manifold M), and g is a fibrewise inner product defined on the distribution Δ. The inner product g is also called a subriemannian metric. An absolutely continuous curve $\gamma : [0, 1] \to M$ on the manifold M is called horizontal if it is almost everywhere tangent to the distribution Δ. Using the inner product g, we can define the length $l(\gamma)$ of a horizontal curve γ by

$$l(\gamma) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt.$$

The subriemannian or Carnot-Caratheodory distance d_{CC} between two points x and y on the manifold M is defined by

$$d_{CC}(x, y) = \inf l(\gamma), \tag{2.1}$$

where the infimum is taken over all horizontal curves which start from x and end at y.

The above distance function may not be well-defined since there may exist two points which are not connected by any horizontal curve. For this we assume that the distribution Δ is bracket generating. Before defining what a bracket generating distribution is, let us introduce several notions. Let Δ_1 and Δ_2 be two distributions on a manifold

M, and let $\mathfrak{X}(\Delta_i)$ be the space of all vector fields contained in the distribution Δ_i. The distribution formed by the Lie brackets of the elements in $\mathfrak{X}(\Delta_1)$ with those in $\mathfrak{X}(\Delta_2)$ is denoted by $[\Delta_1, \Delta_2]$. More precisely

$$[\Delta_1, \Delta_2]_x = \{[w_1, w_2](x) | w_i \in \mathfrak{X}(\Delta_i)\}.$$

We define inductively the following distributions: $[\Delta, \Delta] = \Delta^2$ and $\Delta^k = [\Delta, \Delta^{k-1}]$. A distribution Δ is called *k-generating* if $\Delta^k = TM$ and the smallest such k is called the *degree of nonholonomy.* Finally the distribution is called *bracket generating* if it is k-generating for some k.

Under the bracket generating assumption, the subriemannian distance is well-defined thanks to the following famous Chow-Rashevskii Theorem (see [19] for a proof):

Theorem 2.1. *(Chow-Rashevskii) Assume that the manifold M is connected and the distribution Δ is bracket generating, then there is a horizontal curve joining any two given points.*

If Δ is a bracket generating distribution, then it defines a flag of distribution by

$$\Delta^1 := \Delta \subset \Delta^2 \subset ... \subset TM.$$

If we denote the dimension of the vector space Δ^i_x by n^i_x, then the growth vector of the distribution Δ at the point x is defined by $(n^1_x, n^2_x, ..., n^k_x)$. The distribution Δ is called regular if the growth vector is the same for all points x. The Hausdorff dimension of a subriemannian manifold defined by a regular distribution Δ is given by $\sum_{i=1}^k i(n_i - n_{i-1})$.

As in Riemannian geometry, horizontal curves which realize the infimum in (2.1) are called length minimizing geodesics (or simply geodesics). From now on all manifolds are assumed to be complete with respect to a given subriemannian distance. It means that given any two points on the manifold, there is at least one geodesic joining them. Next we will discuss one type of geodesics called normal geodesics. For this let us recall several notions in the symplectic geometry of the cotangent bundle T^*M. Let $\pi : T^*M \to M$ be the projection map, the tautological one form θ on T^*M is defined by

$$\theta_\alpha(V) = \alpha(d\pi(V)),$$

where α is in the cotangent bundle T^*M and V is a tangent vector on the manifold T^*M at α.

The symplectic two form ω on T^*M is defined as the exterior derivative of the tautological one form: $\omega = d\theta$. It is nondegenerate in the sense that $\omega(V, \cdot) = 0$ if and only if $V = 0$. Given a function $H : T^*M \to \mathbb{R}$ on the cotangent bundle, the Hamiltonian vector field

$\vec{H}$ is defined by $i_{\vec{H}}\omega = -dH$. By the nondegeneracy of the symplectic form ω, the Hamiltonian vector field $\vec{H}$ is uniquely defined.

Given a distribution Δ and a subriemannian metric g on it, we can associate with it a Hamiltonian H on the cotangent bundle T^*M. To do this, let $\alpha : T_xM \to \mathbb{R}$ be a covector in the cotangent space T_x^*M at the point x. The subriemannian metric g defines a bundle isomorphism $I : \Delta^* \to \Delta$ between the distribution Δ and its dual Δ^*. It is defined by

$$g(I(\beta), \cdot) = \beta(\cdot),$$

where β is an element in the dual bundle Δ^* of the distribution Δ.

By restricting the domain of the covector α to the subspace Δ_x of the tangent space T_xM, it defines an element, still called α, in the dual space Δ^*. Therefore, $I(\alpha)$ is a tangent vector contained in the space Δ_x and the Hamiltonian H corresponding to the subriemannian metric g is defined by

$$H(\alpha) := \alpha(I(\alpha)) = g(I(\alpha), I(\alpha)).$$

Note that this construction defines the usual kinetic energy Hamiltonian in the Riemannian case.

Let $\vec{H}$ be the Hamiltonian vector field corresponding to the Hamiltonian H defined above and we denote the corresponding flow by $e^{t\vec{H}}$. If $t \mapsto e^{t\vec{H}}(\alpha)$ is a trajectory of the above Hamiltonian flow, then its projection $t \mapsto \gamma(t) = \pi(e^{t\vec{H}}(\alpha))$ is a locally minimizing geodesic. That means sufficiently short segment of the curve γ is a minimizing geodesic between its endpoints. The minimizing geodesics obtained this way are called normal geodesics. In the special case where the distribution Δ is the whole tangent bundle TM, the distance function (2.1) is the usual Riemannian distance and all geodesics are normal. However, this is not the case for subriemannian manifolds in general. To introduce another class of geodesics, consider the space Ω of horizontal curves with square integrable derivatives. The endpoint map $end : \Omega \to M$ is defined by taking an element γ in space of curves Ω and giving the endpoint $\gamma(1)$ of the curve: $end(\gamma) = \gamma(1)$. Geodesics which are regular points of the endpoint map are automatically normal and those which are critical points are called abnormal. However, there are geodesics which are both normal and abnormal (see [19] and reference therein for more detail about abnormal geodesics).

As an example consider a manifold M of dimension m equipped with a free and proper Lie group action. If G is the group, then the quotient $N := M/G$ is again a manifold of dimension n and the quotient map $\pi_M : M \to N$ defines a principal bundle with a total space M, a base

space N and a structure group G. The kernel of the map $d\pi_M : TM \to TN$ defines a distribution ver of rank $m-n$, called the vertical bundle. A Ehresmann connection is a distribution hor, called horizontal bundle, of rank n which is fibrewise transversal to the vertical bundle ver. The bundle hor is a principal bundle connection (or a connection) if it is preserved under the Lie group action. A subriemannian metric, defined on a connection hor, which is invariant under the Lie group action is called a metric of bundle type. This subriemannian metric descends to a Riemannian metric on the base space N. In this paper two examples will be considered. They are the Heisenberg group and the Hopf fibration.

The Heisenberg group is a principal bundle with the three dimensional Euclidean space $\mathbb{R}^3$ as its total space. If x, y, z are the coordinates of the total space, then the Lie group action is a $\mathbb{R}$-action and it is given by the flow of the vector field ∂_z. The base space of this principal bundle is the two dimensional Euclidean space $\mathbb{R}^2 = \mathbb{R}^3/\mathbb{R}$. The connection hor is defined by the span of the vector fields $\partial_x - \frac{1}{2}y\partial_z$ and $\partial_y + \frac{1}{2}x\partial_z$. The subriemannian metric is defined by declaring that the above vector fields are orthonormal.

The Hopf fibration is a principal S^1-bundle over the two sphere S^2 with the three sphere S^3 as the total space. For explicit formulas let (z_1, z_2) be the coordinates of $\mathbb{C}^2$ and let S^3 be the unit sphere in $\mathbb{C}^2$ defined by $|z_1|^2 + |z_2|^2 = 1$. The map defined by $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2)$ sends the unit sphere S^3 to the unit sphere in the three dimensional Euclidean space $\mathbb{R}^3$. For each point $e^{i\theta}$ in the unit circle, the S^1-action is defined by $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2)$. The vector fields given by $(z_1, z_2) \mapsto (-z_2, z_1)$ and $(z_1, z_2) \mapsto (iz_2, iz_1)$ define a distribution of rank 2 and a subriemannian metric on S^3. The subriemannian metric descends to a Riemannian metric on the 2-sphere S^2 which is twice the usual metric induced from $\mathbb{R}^3$.

3. Generalized Curvatures

In this section we recall the definition of the curvature type invariants studied in [2, 4, 14, 15] and specialize it to the case of a three dimensional contact subriemannian manifold. First let us consider the following general situation. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian and let $\vec{H}$ be the corresponding Hamiltonian vector field. If we denote the flow of the vector field $\vec{H}$ by $e^{t\vec{H}}$ and a point on the manifold T^*M by α, then the differential $de^{-t\vec{H}} : T_{e^{t\vec{H}}(\alpha)}T^*M \to T_\alpha T^*M$

of the map $e^{-t\vec{H}}$ is a symplectic transformation between the symplectic vector spaces $T_{e^{t\vec{H}}(\alpha)}T^*M$ and $T_\alpha T^*M$. Recall that the vertical space V_α at α of the bundle $\pi : T^*M \to M$ is defined as the kernel of the map $d\pi_\alpha : T_\alpha T^*M \to T_{\pi(\alpha)}M$. Since each vertical space V_α is a Lagrangian subspace, the one parameter family of subspaces $t \mapsto J_\alpha(t) := de^{-t\vec{H}}(V_{e^{t\vec{H}}(\alpha)})$ defines a curve of Lagrangian subspaces contained in the symplectic vector space $T_\alpha T^*M$. This curve is called the Jacobi curve at α. In other words if the space of all Lagrangian subspaces, called Lagrangian Grassmannian, of a symplectic vector space Σ is denoted by $LG(\Sigma)$, then the Jacobi curve above is a smooth curve in the Lagrangian Grassmannian $LG(T_\alpha T^*M)$. The Lagrangian Grassmannian is a homogeneous space of the symplectic group, and curvature type invariants of the Hamiltonian H are simply differential invariants of the Jacobi curve under the action of the symplectic group (see [14, 15]).

The construction of differential invariants for a general curve $J(\cdot)$ in the Lagrangian Grassmannian $LG(\Sigma)$ was done in the recent papers [14, 15], though partial results were obtained earlier (see [2, 4]). A principal step is the construction of the canonical splitting:

$$\Sigma = J(t) \oplus \hat{J}(t),$$

where $t \mapsto \hat{J}(t)$ is another curve in the Lagrangian Grassmannian $LG(\Sigma)$ such that $\hat{J}(t)$ is intrinsically defined by the germ of the curve $J(\cdot)$ at time t.

In the case of the Jacobi curve $J_\alpha(\cdot)$, we have the splitting of the symplectic vector space $T_\alpha T^*M$: $T_\alpha T^*M = J_\alpha(t) \oplus \hat{J}_\alpha(t)$. In particular the subspace $J_\alpha(0)$ is the vertical space V_α of the bundle $\pi : T^*M \to M$ and the subspace $\hat{J}_\alpha(0)$ is a complimentary subspace to $J_\alpha(0) = V_\alpha$ at time $t = 0$. Hence, $\{\hat{J}_\alpha(0)\}_{\alpha \in T^*M}$ defines an Ehresmann connection on the bundle $\pi : T^*M \to M$. It is shown (see [2]) that this connection defines a torsion free connection since $\hat{J}_\alpha(0)$ are Lagrangian subspaces of the symplectic vector space $T_\alpha T^*M$. However, it is not a linear connection in general. In the Riemannian case this is, under the identification of the tangent and cotangent spaces by the Riemannian metric, simply the Levi-Civita connection (see [2]).

Using the above splitting we can also define a generalization of the Ricci curvature in the Riemannian geometry. Indeed let $\pi_{J_\alpha(t)}$ and $\pi_{\hat{J}_\alpha(t)}$ be the projections, corresponding to the splitting $T_\alpha T^*M = J_\alpha(t) \oplus \hat{J}_\alpha(t)$, onto the subspaces $J_\alpha(t)$ and $\hat{J}_\alpha(t)$, respectively. Let $w(\cdot)$ be a path contained in the Jacobi curve $J_\alpha(\cdot)$ (i.e. $w(t) \in J_\alpha(t)$). Then

the projection $\pi_{\hat{J}_\alpha(t)}\dot{w}(t)$ of its derivative $\dot{w}(t)$ onto the subspace $\hat{J}_\alpha(t)$ depends only on the vector $w(t)$ but not on the curve $w(\cdot)$. Therefore, it defines a linear operator $\Phi^t_{J_\alpha \hat{J}_\alpha} : J_\alpha(t) \to \hat{J}_\alpha(t)$

$$\Phi^t_{J_\alpha \hat{J}_\alpha}(v) = \pi_{\hat{J}_\alpha(t)} \frac{d}{dt} w(t).$$

Similarly we can also define another operator $\Phi^t_{\hat{J}_\alpha J_\alpha} : \hat{J}_\alpha(t) \to J_\alpha(t)$ by switching the role of J and $\hat{J}$ above. Finally the *generalized Ricci curvature* is defined by negative of the trace of the linear operator $\Phi^0_{\hat{J}_\alpha J_\alpha} \circ \Phi^0_{J_\alpha \hat{J}_\alpha} : J_\alpha(0) \to J_\alpha(0)$.

Recall that a basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ in a symplectic vector space with a symplectic form ω is a Darboux basis if it satisfies $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, and $\omega(f_i, e_j) = \delta_{ij}$. The canonical splitting $\Sigma = J(t) \oplus \hat{J}(t)$ mentioned above is accompanied by a moving Darboux basis $\{e_1(t), ..., e_n(t), f_1(t), ..., f_n(t)\}$ of the symplectic vector space Σ satisfying

$$J(t) = span\{e_1(t), \dots, e_n(t)\},\ \hat{J}(t) = span\{f_1(t), \dots, f_n(t)\}$$

and the structural equations

$$\dot{e}_i(t) = \sum_j (\mathfrak{c}^1_{ij}(t) e_j(t) + \mathfrak{c}^2_{ij}(t) f_j(t)), \quad \dot{f}_i(t) = \sum_j (\mathfrak{c}^3_{ij}(t) e_j(t) + \mathfrak{c}^4_{ij}(t) f_j(t)).$$

This is an analog of the Frenet frame of a curve in the Euclidean space. The generalized Ricci curvature is given by the trace of the matrix $-\mathfrak{c}^3(0)\mathfrak{c}^2(0)$, where $\mathfrak{c}^2(0)$ and $\mathfrak{c}^3(0)$ are the matrices with entries $\mathfrak{c}^2_{ij}(0)$ and $\mathfrak{c}^3_{ij}(0)$, respectively.

The most interesting cases for us are contact subriemannian structures on three dimensional manifolds. To define such a structure, let Δ be a distribution of rank two (i.e. Δ is a vector subbundle of the tangent bundle and the dimension of each fibre is two) on a three dimensional manifold M. We assume that Δ is 2-generating meaning that the vector fields contained the distribution Δ together with their Lie brackets span all tangent spaces of M. In other words

$$TM = span\{X_1, [X_2, X_3] | X_i \in \Delta\}.$$

The structural equations, in this case, have the following form (see appendix 1 for the proof):

Theorem 3.1. *For each fixed α in the manifold T^*M, there is a moving Darboux frame*

$$e_1(t), e_2(t), e_3(t), f_1(t), f_2(t), f_3(t)$$

in the symplectic vector space $T_\alpha T^*M$ and functions R_t^{11}, R_t^{22} of time t such that $\{e_1(t), e_2(t), e_3(t)\}$ form a basis for the Jacobi curve $J_\alpha(t)$ and it satisfies the following structural equations

$$\begin{cases} \dot{e}_1(t) = f_1(t), \\ \dot{e}_2(t) = e_1(t), \\ \dot{e}_3(t) = f_3(t), \\ \dot{f}_1(t) = -R_t^{11} e_1(t) - f_2(t), \\ \dot{f}_2(t) = -R_t^{22} e_2(t), \\ \dot{f}_3(t) = 0. \end{cases}$$

Moreover, $e_3(t) = \frac{1}{\sqrt{2H}}(\vec{E} - t\vec{H})$, $f_3(t) = -\frac{1}{\sqrt{2H}}\vec{H}$, and the generalized Ricci curvature $\mathfrak{Ric}$ is given by $\mathfrak{Ric} = R_0^{11}$.

Next we will write down explicit formulas (Theorem 3.2) for the generalized Ricci curvature $\mathfrak{Ric}$ and the moving Darboux frame $e_1(t)$,$e_2(t)$,$e_3(t)$, $f_1(t)$,$f_2(t)$,$f_3(t)$ in Theorem 3.1.

Let Δ be the contact distribution and let H be the Hamiltonian corresponding to a given subriemannian metric g on Δ. Let σ be an annihilator 1-form of the distribution Δ. That means a vector v is in the distribution Δ if and only if $\sigma(v) = 0$ (i.e. $\ker\sigma = \Delta$). Since Δ is a contact distribution, σ can be chosen in such a way that the restriction of its exterior derivative $d\sigma$ to the distribution Δ is the volume form with respect to the subriemannian metric g. Let $\{v_1, v_2\}$ be a local orthonormal frame in the distribution Δ with respect to the subriemannian metric g and let $v_0 = e$ be the Reeb field defined by the conditions $i_e\sigma = 1$ and $i_e d\sigma = 0$. This defines a convenient frame $\{v_0, v_1, v_2\}$ in the tangent bundle TM and we let $\{\alpha_0 = \sigma, \alpha_1, \alpha_2\}$ be the corresponding dual co-frame in the cotangent bundle T^*M (i.e. $\alpha_i(v_j) = \delta_{ij}$).

The frame $\{v_0, v_1, v_2\}$ and the co-frame $\{\alpha_0, \alpha_1, \alpha_2\}$ defined above induces a frame in the tangent bundle TT^*M of the cotangent bundle T^*M. Indeed, let $\vec{\alpha}_i$ be the vector fields on the cotangent bundle T^*M defined by $i_{\vec{\alpha}_i}\omega = -\alpha_i$. Note that the symbol α_i in the definition of $\vec{\alpha}_i$ represents the pull back $\pi^*\alpha_i$ of the 1-form α on the manifold M by the projection $\pi : T^*M \to M$. This convention of identifying forms in the manifold M and its pull back on the cotangent bundle T^*M will be used for the rest of this paper without mentioning. Let ξ_1 and ξ_2 be the 1-forms defined by $\xi_1 = h_1\alpha_2 - h_2\alpha_1$ and $\xi_2 = h_1\alpha_1 + h_2\alpha_2$, respectively, and let $\vec{\xi}_i$ be the vector fields defined by $i_{\vec{\xi}_i}\omega = -\xi_i$. Finally if we let $h_i : T^*M \to \mathbb{R}$ be the Hamiltonian lift of the vector fields v_i, defined by $h_i(\alpha) = \alpha(v_i)$, then the vector fields $\vec{h}_0, \vec{h}_1, \vec{h}_2, \vec{\sigma}, \vec{\xi}_1, \vec{\xi}_2$ define a local frame for the tangent bundle TT^*M of the cotangent bundle

T^*M. Under the above notation the subriemannian Hamiltonian is given by $H = \frac{1}{2}((h_1)^2 + (h_2)^2)$ and the Hamiltonian vector field is $\vec{H} = h_1\vec{h}_1 + h_2\vec{h}_2$.

We also need the bracket relations of the vector fields v_0, v_1, v_2. Let a_{ij}^k be the functions on the manifold M defined by

$$[v_i, v_j] = a_{ij}^0 v_0 + a_{ij}^1 v_1 + a_{ij}^2 v_2. \tag{3.1}$$

The dual version of the above relation is

$$d\alpha_k = -\sum_{0\le i<j\le 2} a_{ij}^k \alpha_i \wedge \alpha_j. \tag{3.2}$$

By (3.2) and the definition of the Reeb field $e = v_0$, it follows that $d\sigma = d\alpha_0 = \alpha_1 \wedge \alpha_2$. Therefore, $a_{01}^0 = a_{02}^0 = 0$ and $a_{12}^0 = -1$. If we also take the exterior derivative of the equation in (3.2), we get $a_{01}^1 + a_{02}^2 = 0$. Finally we come to the main theorem of this section:

Theorem 3.2. *The Darboux frame $e_1(t), e_2(t), e_3(t), f_1(t), f_2(t), f_3(t)$ and the invariants R_t^{11} and R_t^{22} satisfy $e_i(t) = (e^{t\vec{H}})^* e_i(0)$, $f_i(t) = (e^{t\vec{H}})^* f_i(0)$, $R_t^{11} = (e^{t\vec{H}})^* R_0^{11}$, $R_t^{11} = (e^{t\vec{H}})^* R_0^{11}$, and*

$$\begin{cases} e_2(0) = \frac{1}{\sqrt{2H}}\vec{\sigma}, \\ e_1(0) = \frac{1}{\sqrt{2H}}\vec{\xi}_1, \\ f_1(0) = \frac{1}{\sqrt{2H}}[h_1\vec{h}_2 - h_2\vec{h}_1 + \chi_0\vec{\alpha}_0 + (\vec{\xi}_1 h_{12})\vec{\xi}_1 - h_{12}\vec{\xi}_2], \\ f_2(0) = \frac{1}{\sqrt{2H}}[2H\vec{h}_0 - h_0\vec{H} - \chi_1\vec{\alpha}_0 + (\vec{\xi}_1 a)\vec{\xi}_1 - a\vec{\xi}_2], \\ \mathfrak{Ric} := R_0^{11} = h_0^2 + 2H\kappa - \frac{3}{2}\vec{\xi}_1 a, \\ \mathfrak{r} := R_0^{22} = R_0^{11}\vec{\xi}_1 a - 3\vec{H}\vec{\xi}_1\vec{H}a + 3\vec{H}^2\vec{\xi}_1 a + \vec{\xi}_1\vec{H}^2 a. \end{cases}$$

where

$$\begin{aligned} &a = dh_0(\vec{H}), \\ &\chi_0 = h_2h_{01} - h_1h_{02} + \vec{\xi}_1 a, \\ &\chi_1 = h_0 a + 2\vec{H}\vec{\xi}_1 a - \vec{\xi}_1\vec{H}a, \\ &\kappa = v_1 a_{12}^2 - v_2 a_{12}^1 - (a_{12}^1)^2 - (a_{12}^2)^2 - \tfrac{1}{2}(a_{01}^2 - a_{02}^1). \end{aligned}$$

The rest of this section is devoted to the proof of this theorem. First we need a few lemmas. Let $h_{ij} : T^*M \to \mathbb{R}$ be the Hamiltonian lift of the vector field $[v_i, v_j]$: $h_{ij}(\alpha) = \alpha([v_i, v_j])$. Then the commutator relations of the frame $\{\vec{h}_i, \vec{\alpha}_i | i = 1, 2, 3\}$ is given by the following:

Lemma 3.3.

$$[\vec{h}_i, \vec{h}_j] = \vec{h}_{ij}, \quad [\vec{h}_i, \vec{\alpha}_j] = -\sum_k a_{ik}^j \vec{\alpha}_k, \quad [\vec{\alpha}_i, \vec{\alpha}_j] = 0.$$

Proof. Since the Lie derivative of the symplectic form ω along the Hamiltonian vector field $\vec{h}_i$ vanishes,

$$i_{[\vec{h}_i,\vec{h}_j]}\omega = \vec{h}_i i_{\vec{h}_j}\omega = -d i_{\vec{h}_i} i_{\vec{h}_j}\omega. \tag{3.3}$$

The function $\omega(\vec{h}_i,\vec{h}_j)$ is equal to h_{ij}. Indeed, since $\omega = d\theta$, we have $\theta(\vec{h}_i) = h_i$. By using Cartan's formula, it follows that

$$dh_j(\vec{h}_i) = \omega(\vec{h}_i,\vec{h}_j) = dh_j(\vec{h}_i) - dh_i(\vec{h}_j) - \theta([\vec{h}_i,\vec{h}_j]).$$

Since $d\pi(\vec{h}_i) = v_i$, it implies that $\theta([\vec{h}_i,\vec{h}_j]) = h_{ij}$. Therefore, we have

$$\omega(\vec{h}_i,\vec{h}_j) = -dh_i(\vec{h}_j) = h_{ij}. \tag{3.4}$$

If we combine this with (3.3), the first assertion of the lemma follows.

A calculation similar to the above one shows that

$$i_{[\vec{h}_i,\vec{\alpha}_j]}\omega = \vec{h}_i i_{\vec{\alpha}_j}\omega.$$

By Cartan's formula, the above equation becomes

$$i_{[\vec{h}_i,\vec{\alpha}_j]}\omega = -i_{\vec{h}_i}\pi^* d\alpha_j = -\pi^*(i_{v_i} d\alpha_j).$$

The second assertion follows from this and (3.2).

If we apply Cartan's formula again,

$$i_{[\vec{\alpha}_i,\vec{\alpha}_j]}\omega = \vec{\alpha}_i i_{\vec{\alpha}_j}\omega - i_{\vec{\alpha}_j}\vec{\alpha}_i\omega = -i_{\vec{\alpha}_i} d(\pi^*\alpha_j) + i_{\vec{\alpha}_j} d(\pi^*\alpha_i)$$

Since $d\pi(\vec{\alpha}_i) = 0$, it follows that $i_{[\vec{\alpha}_i,\vec{\alpha}_j]}\omega = 0$. Therefore, the third holds. □

Let $\beta = h_1 dh_2 - h_2 dh_1$, then we also have the following relations:

Lemma 3.4.

$$dh_i(\vec{h}_j) = -h_{ij}, \quad \alpha_i(\vec{h}_j) = -dh_i(\vec{\alpha}_j) = \delta_{ij}, \quad \alpha_i(\vec{\alpha}_j) = 0,$$

$$\beta(\vec{\xi}_2) = dH(\vec{\xi}_1) = 0, \quad \beta(\vec{\xi}_1) = dH(\vec{\xi}_2) = -2H.$$

Proof. The first assertion follows from (3.4) and the last two assertions follow from $d\pi(\vec{h}_i) = v_i$ and $d\pi(\vec{\alpha}_i) = 0$. A computation using $\alpha_i(\vec{h}_j) = \delta_{ij}$ proves the rest of the assertions. □

Proof of Theorem 3.2. If we define E_2 by $E_2(t) = (e^{t\vec{H}})^*\vec{\sigma}$, then $E_2(t)$ is contained in the Jacobi curve $J_\alpha(t)$. Since $e_1(t), e_2(t), e_3(t)$ span the Jacobi curve at time t,

$$E_2(t) = c_1(t)e_1(t) + c_2(t)e_2(t) + c_3(t)e_3(t)$$

for some functions c_i of time t.

Since H is the subriemannian Hamiltonian, the vectors $d\pi(\vec{H}(\alpha))$ are contained in the distribution Δ for each element α in the cotangent bundle T^*M. Therefore, $\omega(\vec{\sigma},\vec{H}) = -\pi^*\sigma(\vec{H}) = 0$. Since $f_3 = -(2H)^{-1/2}\vec{H}$, we have

$$0 = \omega(\vec{\sigma},\vec{H}) = \omega(E_2,\vec{H}) = -(2H)^{-1/2}c_3(t).$$

This shows that $c_3 \equiv 0$ and so $E_2(t) = c_1(t)e_1(t) + c_2(t)e_2(t)$. If we differentiate this with respect to time t, then we have

$$(e^{t\vec{H}})^*[\vec{H},\vec{\sigma}] = \dot{E}_2(t) = \dot{c}_1(t)e_1(t) - c_1(t)f_1(t) + \dot{c}_2(t)e_2(t) - c_2(t)e_1(t).$$

By Cartan's formula, it follows that

$$\omega([\vec{H},\vec{\sigma}],\vec{\sigma}) = \pi^*\sigma([\vec{H},\vec{\sigma}]) = -\pi^*d\sigma(\vec{H},\vec{\sigma}) = 0.$$

By combining this with the above equation for E_2 and $\dot{E}_2$, we have $c_1 \equiv 0$. If we differentiate the equation $E_2(t) = c_2(t)e_2(t)$ with respect to time t again, we get

$$\begin{cases} E_2(t) & = c_2(t)e_2(t) \\ (e^{t\vec{H}})^*(ad_{\vec{H}}(\vec{\sigma})) & = \dot{c}_2(t)e_2(t) + c_2(t)e_1(t) \\ (e^{t\vec{H}})^*(ad^2_{\vec{H}}(\vec{\sigma})) & = \ddot{c}_2(t)e_2(t) + 2\dot{c}_2(t)e_1(t) + c_2(t)f_1(t). \end{cases}$$

This gives $c := 1/c_2(0) = (\omega_\alpha(ad^2_{\vec{H}}(\vec{\sigma}), ad_{\vec{H}}(\vec{\sigma})))^{-1/2}$, and $c_2(t) = (e^{t\vec{H}})^*c_2(0)$. Therefore, $e_2(t) = (e^{t\vec{H}})^*(c\vec{\sigma})$.

To find out what c is more explicitly, we first compute $[\vec{H},\vec{\alpha}_0]$. The Lie bracket is a derivation in each of its entries, so

$$[\vec{H},\vec{\alpha}_0] = [h_1\vec{h}_1 + h_2\vec{h}_2, \vec{\alpha}_0] = -dh_1(\vec{\alpha}_0)\vec{h}_1 - dh_2(\vec{\alpha}_0)\vec{h}_2 + h_1[\vec{h}_1,\vec{\sigma}] + h_2[\vec{h}_2,\vec{\sigma}].$$

It follows from this, Lemma 3.3, and Lemma 3.4 that

$$[\vec{H},\vec{\alpha}_0] = h_1\vec{\alpha}_2 - h_2\vec{\alpha}_1 = \xi_1.$$

Next, we want to compute $[\vec{H},\vec{\xi}_1]$. For this, let

$$[\vec{H},\vec{\xi}_1] = k_0\vec{\alpha}_0 + k_1\vec{\xi}_1 + k_2\vec{\xi}_2 + \sum_{i=0}^{3}\tilde{c}_i\vec{h}_i \tag{3.5}$$

for some functions c_i and k_i.

To compute $\tilde{c}_0$ for instance, we apply α_0 on both sides of (3.5). Using Lemma 3.4 and Cartan's formula, we have $\tilde{c}_0 = 0$. Similar computation gives $\tilde{c}_1 = -h_2$ and $\tilde{c}_2 = h_1$. This shows that

$$[\vec{H},\vec{\xi}_1] = k_0\vec{\alpha}_0 + k_1\vec{\xi}_1 + k_2\vec{\xi}_2 + h_1\vec{h}_2 - h_2\vec{h}_1. \tag{3.6}$$

By applying dh_0 on both sides of (3.6) and using Lemma 3.4 again, we have $k_0 = h_2h_{01} - h_1h_{02} + \vec{\xi}_1 a$. Similar calculations using β and dH give

$$[\vec{H}, \vec{\xi}_1] = h_1\vec{h}_2 - h_2\vec{h}_1 + \chi_0\vec{\alpha}_0 + (\vec{\xi}_1 h_{12})\vec{\xi}_1 - h_{12}\vec{\xi}_2. \tag{3.7}$$

where $\chi_0 = h_2h_{01} - h_1h_{02} + \vec{\xi}_1 a$ and $a = dh_0(\vec{H})$.

It follows that

$$c^{-2} = \omega(ad^2_{\vec{H}}(\vec{\sigma}), ad_{\vec{H}}(\vec{\sigma})) = 2H$$

and $e_2(0) = \frac{1}{\sqrt{2H}}\vec{\alpha}_0$. It also follows from Theorem 3.2 that

$$\begin{aligned} e_1(0) &= \tfrac{1}{\sqrt{2H}}\vec{\xi}_1, \\ f_1(0) &= \tfrac{1}{\sqrt{2H}}[\vec{H}, \vec{\xi}_1], \\ \dot{f}_1(0) &= \tfrac{1}{\sqrt{2H}}[\vec{H}, [\vec{H}, \vec{\xi}_1]], \\ \ddot{f}_1(0) &= \tfrac{1}{\sqrt{2H}}[\vec{H}, [\vec{H}, [\vec{H}, \vec{\xi}_1]]]. \end{aligned} \tag{3.8}$$

A computation similar to that of (3.7) gives

$$[\vec{H}, [\vec{H}, \vec{\xi}_1]] = -2H\vec{h}_0 + h_0\vec{H} + \chi_1\vec{\alpha}_0 + (\chi_2 + \chi_0 - \vec{\xi}_1 a)\vec{\xi}_1 + a\vec{\xi}_2 \tag{3.9}$$

where $\chi_1 = h_0 a + 2\vec{H}\vec{\xi}_1 a - \vec{\xi}_1\vec{H}a$ and $\chi_2 = h_0h_{12} + 2\vec{H}\vec{\xi}_1 h_{12} - \vec{\xi}_1\vec{H}h_{12}$.

It follows from Theorem 3.2, (3.7), and (3.9) that

$$R^{11}_0 = -\chi_0 - \chi_2. \tag{3.10}$$

Since $\dot{f}_1(0) = -R^{11}_0 e_1(0) - f_2(0)$, it follows from (3.8), (3.9), and (3.10) that

$$f_2(0) = \frac{1}{\sqrt{2H}}[2H\vec{h}_0 - h_0\vec{H} - \chi_1\vec{\alpha}_0 + (\vec{\xi}_1 a)\vec{\xi}_1 - a\vec{\xi}_2].$$

A long computation using the bracket relations (3.1) gives

$$\chi_2 = -(h_0)^2 + 2H[(a^1_{12})^2 + (a^2_{12})^2 - v_1a^2_{12} + v_2a^1_{12}] + \vec{\xi}_1 a.$$

and

$$\chi_0 - \frac{1}{2}\vec{\xi}_1 a = h_2h_{01} - h_1h_{02} + \frac{1}{2}\vec{\xi}_1 a = H(a^2_{01} - a^1_{02}).$$

It follows as claimed that

$$R^{11}_0 = h_0^2 + 2H\kappa - \frac{3}{2}\vec{\xi}_1 a.$$

To prove the formula for R^{22}, we differentiate the equation $\dot{f}_1(t) = -R^{11}_t e_1(t) - f_2(t)$ and combine it with the equation $\dot{f}_2(t) = -R^{22}_t e_2(t)$. We have

$$R^{22}_0 e_2(0) = \ddot{f}_1(0) + \vec{H} R^{11}_0 e_1(0) + R^{11}_0 f_1(0).$$

Therefore, by applying dh_0 on both sides and using $dh_0(e_1(0)) = 0$, we get

$$R^{22}_0 = -\sqrt{2H}[dh_0(\ddot{f}_1(0)) + R^{11}_0 dh_0(f_1(0))].$$

By using Cartan's formula and (3.8), it follows that

$$\sqrt{2H} dh_0(f_1(0)) = dh_0([\vec{H}, \vec{\xi}_1]) = -\vec{\xi}_1 a,$$

$$\sqrt{2H} dh_0(\dot{f}_1(0)) = dh_0([\vec{H}, [\vec{H}, \vec{\xi}_1]]) = \vec{\xi}_1 \vec{H} a - 2\vec{H}\vec{\xi}_1 a,$$

and therefore,

$$\sqrt{2H} dh_0(\ddot{f}_1(0)) = 3\vec{H}\vec{\xi}_1\vec{H} a - 3\mathcal{L}^2_{\vec{H}}\vec{\xi}_1 a - \vec{\xi}_1\vec{H}^2 a.$$

The formula for R^{22}_0 follows from this.

□

4. Measure Contraction Properties

Measure contraction property is introduced in [25] as one of the generalizations of curvature dimension bound to all metric measure spaces. In the setting of a subriemannian manifold with a 2-generating distribution, a simpler definition can be given. To do this, we first recall the recent results on the theory of optimal transportation in [3] and [10]. Let μ and ν be two Borel probability measures on the subriemannian manifold M with a distribution Δ and a subriemannian metric g. If we let H be the Hamiltonian corresponding to the metric g and d_{CC} be the corresponding subriemannian distance, then the optimal transportation problem is the following minimization problem:

Find a Borel map $\varphi : M \to M$ which achieves the following infimum

$$\inf \int_M d^2_{CC}(x, \varphi(x)) d\mu(x) \tag{4.1}$$

where the infimum is taken over all Borel map φ which pushes μ forward to ν: $\varphi_{t*}\mu = \nu$ (i.e. $\mu(\varphi^{-1}(U)) = \nu(U)$ for all Borel sets U).

The minimizers to the above problem are called optimal maps. The following theorem is one of the main results in [3] which generalizes the earlier work in [7, 18].

Theorem 4.1. *Assume that the distribution Δ is 2-generating and the measure μ is absolutely continuous with respect to the Lebesgue measure, then the optimal transportation problem has a solution φ and any optimal map equals to this one μ almost everywhere. Moreover, φ is given by $\varphi(x) = \pi(e^{1\vec{H}}(-df_x))$ for some Lipschitz function f.*

Many important results in the theory of optimal transportation rely on the study of the following 1-parameter family of maps called *displacement interpolation* introduced by R. McCann (see [27] for the history and importance of displacement interpolation):

$$\varphi_t(x) := \pi(e^{t\vec{H}}(-df_x)).$$

If φ_1 is the optimal map between the measures μ and ν, then φ_t is optimal between μ and $\varphi_{t*}\mu$. All the above results also hold when the distance squared cost d^2 is replaced by costs defined by Lagrange's problem (see [5, 3]). In those cases the subriemannian Hamiltonian H in Theorem 4.1 is replaced by more general Hamiltonians.

If we set, in the displacement interpolation, $f(x) = d^2_{CC}(x, x_0)$ for some given point x_0 on the manifold M, then φ_1 is the optimal map which pushes any measure μ to the delta mass concentrated at a point x_0. In this case the curves defined by $t \mapsto \varphi_t(x) := \pi(e^{t\vec{H}}(-df_x))$ are unique normal geodesics joining the points x and x_0 for Lebesgue almost all x.

Let η be a Borel measure on the manifold M and let φ_t be the map $\varphi_t(x) = \pi(e^{t\vec{H}}(-df_x))$, where $f(x) = d^2_{CC}(x, x_0)$. The metric measure space (M, d_{CC}, η) satisfies the measure contraction property $MCP(k, N)$ if

$$(\mathfrak{S}_t \circ \varphi_t)\, \eta(U) \leq \eta(\varphi_t(U))$$

for each η measurable set U and each point x_0 in the manifold M, where

$$\mathfrak{S}_t = (1-t)\left(\frac{\sin K_0}{\sin K_t}\right)^{N-1}$$

if $k > 0$,

$$\mathfrak{S}_t = (1-t)\left(\frac{\sinh K_0}{\sinh K_t}\right)^{N-1}$$

if $k < 0$,

$$\mathfrak{S}_t = (1-t)^N$$

if $k = 0$, and

$$K_t(x) = \sqrt{\frac{|k|}{N-1}}\frac{d_{CC}(x_0, x)}{1-t}.$$

Next we specialize to the case where M is a three dimensional manifold with a contact distribution Δ and a subriemannian metric g. Let d_{CC} be the corresponding subriemannian distance function and let R^{11}, R^{22} be the invariants defined as in Theorem 3.1. Recall that the kernel of the map $d\pi : TT^*M \to TM$ defines the vertical bundle V on the manifold T^*M. Let $\mathfrak{m}$ be the three form on the manifold T^*M such that it is zero on the vertical spaces V_α and $\mathfrak{m}_\alpha(f_1(0), f_2(0), f_3(0)) = 1$ where $\{f_1(0), f_2(0), f_3(0)\}$ is defined as in Theorem 3.1. The following lemma shows that the Hamiltonian H is unimodular (see Appendix for the definition of unimodular).

Lemma 4.2. *Let η be a smooth volume form on the manifold M such that $\eta(e, v_1, v_2) = 1$, then $\pi^*\eta = \sqrt{2H}\mathfrak{m}$.*

Proof. Clearly, $\pi^*\eta$ is zero on the space V. Therefore, it is enough to show that $\pi^*\eta(f_1(0), f_2(0), f_3(0)) = \sqrt{2H}$, and this follows from Theorem 3.2 and the definition of η. □

We also use the same notation for the measure corresponding to the volume form η. Recall that $\mathfrak{R}ic$ and $\mathfrak{r}$ denote the invariants defined in Theorem 3.2. If we let U be a subset of the manifold M and we fix a point x_0 in M, then we denote by $I(x_0, U)$ all the covectors along all geodesics joining from any point in U to the point x_0. More precisely, $I(x_0, U)$ is the set of all covectors of the form $e^{t\vec{H}}(\alpha_x)$ contained in T^*M, where x is a point in U, t is in $[0, 1]$, and the curve $t \mapsto \pi(e^{t\vec{H}}(\alpha_x))$ is a minimizer joining the point x and x_0. Finally we come to the main result:

Theorem 4.3. *If $\mathfrak{Ric} := R_0^{11} \geq 2rH$ and $\mathfrak{r} := R_0^{22} \geq 0$ on $I(x_0, U)$, then*

$$(\mathfrak{S}_t \circ \varphi_t)\,\eta(U) \leq \eta(\varphi_t(U)), \tag{4.2}$$

where

$$\mathfrak{S}_t = \frac{(1-t)(2 - 2\cos\mathcal{T}_0 - \mathcal{T}_0 \sin\mathcal{T}_0)}{(2 - 2\cos\mathcal{T}_t - \mathcal{T}_t \sin\mathcal{T}_t)} \tag{4.3}$$

if $r > 0$,

$$\mathfrak{S}_t = \frac{(1-t)(2 - 2\cosh\mathcal{T}_0 + \mathcal{T}_0 \sinh\mathcal{T}_0)}{(2 - 2\cosh\mathcal{T}_t + \mathcal{T}_t \sinh\mathcal{T}_t)} \tag{4.4}$$

if $r < 0$,

$$\mathfrak{S}_t = (1-t)^5 \tag{4.5}$$

if $r = 0$*, and*

$$\mathcal{T}_t(x) = \frac{\sqrt{|r|}d_{CC}(x_0, x)}{1-t}.$$

Definition 4.4. We say that a metric measure space satisfies the *generalized measure contraction property* $MCP(r; 2, 3)$ if (4.2) holds for each η-measurable set U and each x_0 in the manifold M.

The following theorem is an immediate consequence of Theorem 4.3 and Definition 4.4.

Theorem 4.5. *If* $\mathfrak{Ric} \geq 2rH$ *and* $\mathfrak{r} = 0$*, then the metric measure space* (M, d_{CC}, η) *satisfies the generalized measure contraction property* $MCP(r; 2, 3)$*. In particular if* $r \geq 0$*, then* (M, d_{CC}, η) *satisfies the measure contraction property* $MCP(0, 5)$*.*

Remark 4.6. It is I. Zelenko's observation that if $\mathfrak{r}$ is bounded below globally, then it is in fact identically zero.

Remark 4.7. In general, the inequality in Theorem 4.3 is very sensitive to the location of the set U and the point x_0 due to the anisotropy of the subriemannian metric. This is an important aspect which will be investigated in a further work.

Remark 4.8. The pair $(2, 3)$ in the generalized measure contraction property is the growth vector of the three dimensional contact subriemannian manifold. The growth vector of the n-dimensional Riemannian manifold is n. In general the growth vector determines the Hausdorff dimension (see Section 2) and this should shape the measure contraction property. In this paper we add $MCP(r; 2, 3)$ to the measure contraction property $MCP(r; n)$ found earlier by Sturm. It would be very interesting to find appropriate measure contraction properties for other growth vectors. We also remark that the ingredients used in the proof of Theorem 4.5 are also present in higher dimensions. This includes the recent result in [14, 15], a comparison principle of matrix Riccati equations, and the solvability of matrix Riccati equations with constant coefficients. Therefore, the proof works for more general subriemannian manifolds without abnormal minimizers.

As a corollary of Theorem 4.5, we have the following doubling property (see [25]).

Corollary 4.9. *(Doubling) Let* $B_x(r)$ *be the ball of radius* r *centered at* x *in the space* (M, d_{CC}, η)*. If* $\mathfrak{Ric} \geq 0$ *and* $\mathfrak{r} = 0$*, then it satisfies the following doubling property:*

$$\eta(B_x(2r)) \leq 2^5 \eta(B_x(r)).$$

Recall that a Borel function $h : M \to \mathbb{R}$ is the upper gradient of a function $f : M \to \mathbb{R}$ if

$$|f(x(0)) - f(x(1))| \leq l(x(\cdot)) \int_0^1 h(x(s))ds$$

for each curve $x(\cdot)$ of finite length $l(x(\cdot))$.

The following local Poincaré inequality also holds as a corollary of Theorem 4.5 (see the proof of [17, Theorem 3.1] and [17, Theorem 2.5]).

Corollary 4.10. *(Local Poincaré Inequality) If the manifold M is compact, $\mathfrak{Ric} \geq 0$, and $\mathfrak{r} = 0$, then (M, d_{CC}, η) satisfies the following local Poincaré inequality*

$$\frac{1}{\nu(B_x(r))} \int_{B_x(r)} |f(x) - \langle f \rangle_{B_x(r)}| d\eta(x) \leq \frac{Cr}{\nu(B_x(2r))} \int_{\nu(B_x(2r))} h(x) d\eta(x),$$

for some constant C and where $\langle f \rangle_{B_x(r)} = \frac{1}{\eta(B_x(r))} \int_{B_x(r)} f(x) d\eta(x)$.

The rest of this section is devoted to the proof of Theorem 4.5.

Proof of Theorem 4.5. From the main result in [8], the function $f(x) = d(x, x_0)$ is locally semiconcave on $M - \{x_0\}$. So, by [10, Theorem 3.5] and [10, Section 3.4], the measures $\varphi_{t*}\eta$ are absolutely continuous with respect to the Lebesgue class. That means $\varphi_{t*}\eta = \rho_t\eta$ for some function ρ_t.

The function $f(x) = d(x, x_0)$ is locally semiconcave on $M - \{x_0\}$, so it is twice differentiable almost everywhere by Alexandrov's theorem (see for instance [27]). If we denote the differential of the map $x \mapsto -df_x$ by $\mathcal{F}$, then $d\varphi_t = d\pi de^{t\vec{H}}\mathcal{F}$. Let $e_i(t)$ and $f_i(t)$ be the Darboux frame at α defined as in Theorem 3.1 and let $\varsigma_i = d\pi(f_i(0))$. Then the vectors $\{\mathcal{F}(\varsigma_1), \mathcal{F}(\varsigma_2), \mathcal{F}(\varsigma_3)\}$ span a linear subspace W of $T_\alpha T^*M$. Let $e_i(t)$ and $f_i(t)$ be the Darboux frame at α defined as in Theorem 3.1, then $\mathcal{F}(\varsigma_i)$ can be written as

$$\mathcal{F}(\varsigma_i) = \sum_{k=1}^{3} (a_{ij}(t)e_j(t) + b_{ij}(t)f_j(t)) \quad \text{or} \quad \Psi = A_tE_t + B_tF_t,$$

where A_t is the matrix with entries $a_{ij}(t)$, B_t is the matrix with entries $b_{ij}(t)$, and Ψ, E_t, and F_t are matrices with rows $\mathcal{F}(\varsigma_i)$, $e_i(t)$, and $f_i(t)$, respectively.

It follows from absolute continuity of the measure $\varphi_{t*}\eta$ and the result in [3, 10] that the map $\varphi_t(x)$ is injective for η almost all x. We fix a point z for which the map φ_t is injective and the path $s \mapsto \varphi_s(z)$ is minimizing. Such a point exists Lebesgue and hence

η almost everywhere. It follows from [1, Theorem 1.2] that there is no conjugate point along the curve $s \mapsto \varphi_s(z)$. Therefore, the map φ_s is nonsingular for each s in $[0,t]$ and so $\rho(\varphi_t(z)) \neq 0$ for each s in $[0,t]$. Let S_t be the matrix defined as in Theorem 7.2. Recall that S_t is defined as follow: the linear space W is transversal to the space $J_\alpha(t) = span\{e_1(t), ..., e_n(t)\}$. Therefore, the linear subspace W defined above is the graph of a linear map from the space $span\{f_1(t), ..., f_n(t)\}$ to the space $J_\alpha(t) = span\{e_1(t), ..., e_n(t)\}$. Let S_t be the corresponding matrix (i.e. the linear map is given by $f_i(t) \mapsto \sum_{i=1}^3 S_t^{ij} e_j(t)$, where S_t^{ij} are the entries of the matrix S_t). Finally recall that S_t satisfies $S_t = B_t^{-1} A_t$.

Using the structural equation (3.1) and Theorem 7.2, we get the following.

Lemma 4.11. *The matrices S_t satisfies the following matrix Riccati equation:*

$$\dot{S}_t - R_t + S_t C_1 + C_1^T S_t - S_t C_2 S_t = 0,$$

where $R_t = \begin{pmatrix} R_t^{11} & 0 & 0 \\ 0 & R_t^{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

If t is sufficiently close to 1, then S_t^{-1} exists and it is the solution to the following initial value problem

$$\frac{d}{dt}(S_t^{-1}) + S_t^{-1} R_t S_t^{-1} - C_1 S_t^{-1} - S_t^{-1} C_1^T + C_2 = 0 \quad \textit{and} \quad S_1^{-1} = 0.$$

Proof. The matrix Riccati equation follows from (3.1) and Theorem 7.2. To show that $S_1^{-1} = 0$, it is enough to show that $B_1 = 0$. If we let γ be a path such that $\dot{\gamma}(0) = \varsigma_i$, then $\varphi_1(\gamma(s)) = x_0$. By differentiating this equation with respect to s, we get

$$d\pi de^{1 \cdot \vec{H}} \mathcal{F}(\varsigma_i) = d\varphi_1(\varsigma_i) = 0.$$

It follows that $\mathcal{F}(\varsigma_i)$ is contained in $span\{e_1(1), e_2(1), e_3(1)\}$ and so $B_1 = 0$. $\square$

Let us consider the following simpler matrix Ricatti equation:

$$\frac{d}{dt}(\tilde{S}_t^{-1}) + \tilde{S}_t^{-1} \tilde{R}_t \tilde{S}_t^{-1} - C_1 \tilde{S}_t^{-1} - \tilde{S}_t^{-1} C_1^T + C_2 = 0 \tag{4.6}$$

together with the condition $\tilde{S}_1^{-1} = 0$, where $\tilde{R} = \begin{pmatrix} 2rH & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Lemma 4.12. *Let $\tau_t = (1-t)\sqrt{2\,|r|\,H}$. If $r>0$, then the solution to (4.6) is given by*

$$\tilde{S}_t = \begin{pmatrix} \frac{\tau_0(\sin\tau_t - \tau_t\cos\tau_t)}{\mathcal{D}} & \frac{\tau_0^2(1-\cos\tau_t)}{\mathcal{D}} & 0 \\ \frac{\tau_0^2(1-\cos\tau_t)}{\mathcal{D}} & \frac{\tau_0^3\sin\tau_t}{\mathcal{D}} & 0 \\ 0 & 0 & \frac{1}{1-t} \end{pmatrix}$$

where $\mathcal{D} = 2 - 2\cos\tau_t - \tau_t\sin\tau_t$.

If $r<0$, then it is

$$\tilde{S}_t = \begin{pmatrix} \frac{\tau_0(\tau_t\cosh\tau_t - \sinh\tau_t)}{\mathcal{D}^h} & \frac{\tau_0^2(\cosh\tau_t - 1)}{\mathcal{D}^h} & 0 \\ \frac{\tau_0^2(\cosh\tau_t-1)}{\mathcal{D}^h} & \frac{\tau_0^3\sinh\tau_t}{\mathcal{D}^h} & 0 \\ 0 & 0 & \frac{1}{1-t} \end{pmatrix}$$

where $\mathcal{D}^h = 2 - 2\cosh\tau_t + \tau_t\sinh\tau_t$.

Finally if $r=0$, then the solution becomes

$$\tilde{S}_t = \frac{1}{(1-t)^3}\begin{pmatrix} 4(1-t)^2 & 6(1-t) & 0 \\ 6(1-t) & 12 & 0 \\ 0 & 0 & (1-t)^2 \end{pmatrix}.$$

Proof of Lemma 4.12. In the case $r=0$, there is no quadratic term in the matrix Ricatti equation. Therefore, the proof in this case is straightforward and will be omitted.

For other values of r, consider the matrix $\mathcal{A} = \begin{pmatrix} C_1 & -C_2 \\ \tilde{R} & -C_1^T \end{pmatrix}$ and the corresponding matrix differential equation $\frac{d}{dt}q = \mathcal{A}q$ together with the condition $q(1) = I$.

The fundamental solution is given by

$$q(t) = e^{(t-1)\mathcal{A}} = \begin{pmatrix} \cos\tau_t & 0 & 0 & \frac{\sin\tau_t}{\tau_0} & \frac{1-\cos\tau_t}{\tau_0^2} & 0 \\ -\frac{\sin\tau_t}{\tau_0} & 1 & 0 & \frac{\cos\tau_t - 1}{\tau_0^2} & \frac{\sin\tau_t - \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1-t \\ -\tau_0\sin\tau_t & 0 & 0 & \cos\tau_t & \frac{\sin\tau_t}{\tau_0} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

if $r > 0$ and it is

$$q(t) = e^{(t-1)\mathcal{A}} = \begin{pmatrix} \cosh \tau_t & 0 & 0 & \frac{\sinh \tau_t}{\tau_0} & \frac{\cosh \tau_t - 1}{\tau_0^2} & 0 \\ -\frac{\sinh \tau_t}{\tau_0} & 1 & 0 & \frac{1-\cosh \tau_t}{\tau_0^2} & \frac{\tau_t - \sinh \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1-t \\ \tau_0 \sinh \tau_t & 0 & 0 & \cosh \tau_t & \frac{\sinh \tau_t}{\tau_0} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

if $r < 0$.

It follows from [13, Theorem 1] that

$$\begin{aligned} S_t^{-1} &= \begin{pmatrix} \frac{\sin \tau_t}{\tau_0} & \frac{1-\cos \tau_t}{\tau_0^2} & 0 \\ \frac{\cos \tau_t - 1}{\tau_0^2} & \frac{\sin \tau_t - \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1-t \end{pmatrix} \begin{pmatrix} \cos \tau_t & \frac{\sin \tau_t}{\tau_0} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\tan \tau_t}{\tau_0} & \frac{\cos \tau_t - 1}{\tau_0^2 \cos \tau_t} & 0 \\ \frac{\cos \tau_t - 1}{\tau_0^2 \cos \tau_t} & \frac{\tan \tau_t - \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1-t \end{pmatrix}. \end{aligned}$$

if $r > 0$ and

$$\begin{aligned} S_t^{-1} &= \begin{pmatrix} \frac{\sinh \tau_t}{\tau_0} & \frac{\cosh \tau_t - 1}{\tau_0^2} & 0 \\ \frac{1-\cosh \tau_t}{\tau_0^2} & \frac{\tau_t - \sinh \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1-t \end{pmatrix} \begin{pmatrix} \cosh \tau_t & \frac{\sinh \tau_t}{\tau_0} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\tanh \tau_t}{\tau_0} & \frac{1-\cosh \tau_t}{\tau_0^2 \cosh \tau_t} & 0 \\ \frac{1-\cosh \tau_t}{\tau_0^2 \cosh \tau_t} & \frac{\tau_t - \tanh \tau_t}{\tau_0^3} & 0 \\ 0 & 0 & 1-t \end{pmatrix}. \end{aligned}$$

if $r < 0$.

Therefore, inverting the above matrix gives the result. □

It follows from the assumption on the set U that $R_t^{11} \geq 2rH$ and $R_t^{22} \geq 0$. Therefore, by comparison theorem of the matrix Riccati equation (see [11, Theorem 2.1]), we have $S_t^{-1} \geq \tilde{S}_t^{-1} \geq 0$ for t close enough to 1. Here $A \geq B$ means that $A - B$ is nonnegative definite. By monotonicity (see [6, Proposition V.1.6]), $0 \leq S_t \leq \tilde{S}_t$ for t close enough to 1. If we apply the same comparison principle to S_t and $\tilde{S}_t$, then $0 \leq S_t \leq \tilde{S}_t$ for all t in $[0, 1]$. Therefore,

$$tr\left(\tilde{S}_t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \geq tr\left(S_t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \tag{4.7}$$

If $r=0$, then

$$\tilde{S}_t^{11}+\tilde{S}_t^{33}=\frac{5}{1-t}.$$

If $r>0$, then

$$\tilde{S}_t^{11}+\tilde{S}_t^{33}=\frac{\tau_0(\sin\tau_t-\tau_t\cos\tau_t)}{2-2\cos\tau_t-\tau_t\sin\tau_t}+\frac{1}{1-t}.$$

If $r<0$, then

$$\tilde{S}_t^{11}+\tilde{S}_t^{33}=\frac{\tau_0(\tau_t\cosh\tau_t-\sinh\tau_t)}{2-2\cosh\tau_t+\tau_t\sinh\tau_t}+\frac{1}{1-t}.$$

If we integrate the above equations, we get

$$\int_0^t \tilde{S}_s^{11}+\tilde{S}_s^{33}ds=-\log(1-t)^5. \tag{4.8}$$

if $r=0$,

$$\int_0^t (\tilde{S}_s^{11}+\tilde{S}_s^{33})ds=-\log\left[\frac{(1-t)(2-2\cos\tau_t-\tau_t\sin\tau_t)}{(2-2\cos\tau_0-\tau_0\sin\tau_0)}\right]. \tag{4.9}$$

if $r>0$, and

$$\int_0^t (\tilde{S}_s^{11}+\tilde{S}_s^{33})ds=-\log\left[\frac{(1-t)(2-2\cosh\tau_t+\tau_t\sinh\tau_t)}{(2-2\cosh\tau_0+\tau_0\sinh\tau_0)}\right]. \tag{4.10}$$

if $r<0$.

It follows from Theorem 7.2 that

$$\rho_t(\varphi_t(z))=e^{\int_0^t S_s^{11}+S_s^{33}ds}.$$

If we combine this with (4.7), (4.8), (4.9), and (4.10), then

$$\left[\frac{(1-t)(2-2\cos\tau_t-\tau_t\sin\tau_t)}{(2-2\cos\tau_0-\tau_0\sin\tau_0)}\right]\rho_t(\varphi_t(z))\le 1$$

if $r>0$,

$$\left[\frac{(1-t)(2-2\cosh\tau_t+\tau_t\sinh\tau_t)}{(2-2\cosh\tau_0+\tau_0\sinh\tau_0)}\right]\rho_t(\varphi_t(z))\le 1$$

if $r<0$, and

$$(1-t)^5\rho_t(\varphi_t(z))\le 1$$

if $r=0$.

To complete the proof of the theorem, note that the above inequalities hold η-almost everywhere in the set U and $\tau_t=(1-t)\sqrt{2\,|r|\,H}=(1-t)\sqrt{|r|}d_{CC}(x_0,z)=\sqrt{|r|}d_{CC}(x_0,\varphi_t(z))$. It follows that

$$(\mathfrak{S}_t\circ\varphi_t)\,\rho_t\circ\varphi_t\le 1$$

η-almost everywhere on U. Therefore,

$$\int_U (\mathfrak{S}_t \circ \varphi_t)\, d\eta = \int_{\varphi_t(U)} \mathfrak{S}_t\, d(\varphi_{t*}\eta) = \int_{\varphi_t(U)} \mathfrak{S}_t \rho_t d\eta \leq \int_{\varphi_t(U)} d\eta.$$

This completes the proof. $\square$

5. Isoperimetric problems

In this section we specialize to the case which model the isoperimetric problem or a particle in a constant magnetic field on a Riemannian surface. More precisely, assume that the vector field e, which is transversal to the distribution Δ, defines a free and proper Lie group G-action (i.e. $G = S^1$ or $G = \mathbb{R}$). Then the quotient $N := M/G$ is again a manifold. Assume also that the subriemannian metric g is a metric of bundle type (i.e g is invariant under the above action). Under these assumptions the subriemannian metric g descends to a Riemannian metric on the surface N. In this case Theorem 3.2 and 4.5 simplify to

Theorem 5.1.

$$\begin{cases} e_2(0) = \frac{1}{\sqrt{2H}}\vec{\sigma}, \\ e_1(0) = \frac{1}{\sqrt{2H}}\vec{\xi}_1, \\ f_1(0) = \frac{1}{\sqrt{2H}}[h_1\vec{h}_2 - h_2\vec{h}_1 + 2Ha_{01}^2\vec{\alpha}_0 + (\vec{\xi}_1 h_{12})\vec{\xi}_1 - h_{12}\vec{\xi}_2], \\ f_2(0) = \frac{1}{\sqrt{2H}}[2H\vec{h}_0 - h_0\vec{H}], \\ \mathfrak{Ric} = h_0^2 + 2H\kappa, \\ \mathfrak{r} = 0. \end{cases}$$

where κ is the Gauss curvature of the surface N.

As a consequence the metric measure space (M, d_{CC}, η) satisfies the generalized measure contraction property $MCP(\kappa; 2, 3)$. In particular, it satisfies the measure contraction property $MCP(0, 5)$ if $\kappa \geq 0$.

Proof. Since g is a metric of bundle type, the following holds.

Lemma 5.2. *Under the above assumptions, the functions a_{ij}^k in the bracket relation (3.1) satisfies*

$$a_{01}^0 = a_{02}^0 = a_{01}^1 = a_{02}^2 = 0 \text{ and } a_{01}^2 = -a_{02}^1.$$

Proof of Lemma 5.2. If the flow of the vector field $e = v_0$ is denoted by e^{te}, then the invariance of the subriemannian metric under the group action implies that

$$g((e^{te})^* v_i, (e^{te})^* v_j) = \delta_{ij}, \quad \sigma((e^{te})^* v_j) = 0, \quad i, j = 1, 2.$$

By differentiating the above equations with respect to time t, it follows that

$$\alpha_j([e,v_i]) + \alpha_i([e,v_j]) = 0, \quad \sigma([e,v_j]) = 0, \quad i,j = 1,2.$$

If we apply the bracket relations (3.1) of the frame v_1, v_2, v_3, we have

$$a_{0i}^j + a_{0j}^i = \alpha_j([e,v_i]) + \alpha_i([e,v_j]) = 0, \quad a_{0j}^0 = \sigma([e,v_j]) = 0, \quad i,j = 1,2.$$

□

It follows that

Lemma 5.3. *The function h_0 is a constant of motion of the flow $e^{t\vec{H}}$. i.e. $a = dh_0(\vec{H}) = 0$.*

Proof of Lemma 5.3. This follows from general result in Hamiltonian reduction. In this special case this can also be seen as follow. By Lemma 3.4

$$dh_0(\vec{H}) = dh_0(h_1\vec{h}_1 + h_2\vec{h}_2) = h_1h_{10} + h_2h_{20}. \tag{5.1}$$

By Lemma 5.2 we also have

$$h_{10} = -a_{01}^0 h_0 - a_{01}^1 h_1 - a_{01}^2 h_2 = -a_{01}^2 h_2.$$

Similarly $h_{20} = -a_{02}^1 h_1$. The result follows from this, (5.1), and Lemma 5.2. □

It follows from Lemma 5.2 and Lemma 5.3 that $\chi_0 = 2Ha_{01}^2$. It remains to show that κ is the Gauss curvature of the surface N. By Lemma 5.2 κ simplifies to

$$\kappa = v_1 a_{12}^2 - v_2 a_{12}^1 - (a_{12}^1)^2 - (a_{12}^2)^2. \tag{5.2}$$

Let w_1 and w_2 be a local orthonormal frame on the surface N. Let $\tilde{w}_1$ and $\tilde{w}_2$ be the horizontal lift of w_1 and w_2, respectively. Since $\tilde{w}_1$ and $\tilde{w}_2$ are orthonormal with respect to the subriemannian metric, we can set $v_i = \tilde{w}_i$. It follows from (3.1) that $[\tilde{w}_1, \tilde{w}_2] = a_{12}^1\tilde{w}_1 + a_{12}^2\tilde{w}_2$. Let us denote the covariant derivative on the Riemannian manifold N by ∇. It follows from Koszul formula ([22, Theorem 3.11]) that

$$\begin{aligned} \nabla_{v_1} v_1 = -a_{12}^1 v_2, \quad \nabla_{v_2} v_2 = -a_{12}^2 v_1, \\ \nabla_{v_1} v_2 = a_{12}^1 v_1, \quad \nabla_{v_2} v_1 = -a_{12}^2 v_2. \end{aligned} \tag{5.3}$$

Since the covariant derivative ∇ is tensorial in the bottom slot and is a derivation in the other slot, it follows from (5.3) that

$$\begin{aligned} \nabla_{[v_1,v_2]} v_1 &= \nabla_{a_{12}^1 v_1 + a_{12}^2 v_2} v_1 \\ &= a_{12}^1 \nabla_{v_1} v_1 + a_{12}^2 \nabla_{v_2} v_1 \\ &= -[(a_{12}^1)^2 + (a_{12}^2)^2] v_2 \end{aligned}$$

and

$$\begin{aligned}[\nabla_{v_1},\nabla_{v_2}]v_1 &= \nabla_{v_1}\nabla_{v_2}v_1-\nabla_{v_2}\nabla_{v_1}v_1\\ &= -\nabla_{v_1}(a_{12}^2v_2)+\nabla_{v_2}(a_{12}^1v_2)\\ &= -(v_1a_{12}^2)v_2+(v_2a_{12}^1)v_2-2a_{12}^1a_{12}^2v_1.\end{aligned}$$

Therefore, it follows from the above calculation and (5.2) that the Gauss curvature is given by

$$\begin{aligned}\kappa &=< \nabla_{[v_1,v_2]}v_1-[\nabla_{v_1},\nabla_{v_2}]v_1, v_2 >\\ &= -(a_{12}^1)^2-(a_{12}^2)^2+v_1a_{12}^2-v_2a_{12}^1.\end{aligned}$$

as claimed. □

Recall that the Heisenberg group is the Euclidean space $\mathbb{R}^3$ equipped with the distribution $\Delta=\text{span}\{v_1=\partial_x-\frac{1}{2}y\partial_z, v_2=\partial_y+\frac{1}{2}x\partial_z\})$. Let g be the subriemannian metric for which v_1 and v_2 is orthonormal and let H be the subriemannian Hamiltonian. The vector e which defines the action is given by $e=[v_1,v_2]=\partial_z$. This defines a $\mathbb{R}$-action and the quotient of the manifold M by this action is $N=M/G=\mathbb{R}^2$. The measure η is the Lebesgue measure. Therefore, by applying Theorem 4.5 and Theorem 5.1, we recover the following theorem of [12].

Theorem 5.4. *The Heisenberg group with subriemannian metric defined above together with the Lebesgue measure satisfies the measure contraction property $MCP(0,5)$.*

Next we look at the Hopf fibration. Let S^3 be the unit sphere in $\mathbb{C}^2$. The vector field $(z_1,z_2)\mapsto(iz_1,-iz_2)$ defines a circle S^1 action on S^3. The quotient $N=M/G$ is the unit 2-sphere S^2. The vector fields given by $(z_1,z_2)\mapsto(-z_2,z_1)$ and $(z_1,z_2)\mapsto(iz_2,iz_1)$ define a distribution of rank 2 and a subriemannian metric on S^3.

Theorem 5.5. *The 3-sphere S^3 equipped with the above subriemannian metric satisfies the generalized measure contraction property $MCP(2;2,3)$. In particular, it satisfies the measure contraction property $MCP(0,5)$.*

6. Appendix 1: Proof of Theorem 3.1

According to the main result in [14, 15], there exists a family of Darboux frame $\{e_1(t),e_2(t),e_3(t),f_1(t),f_2(t),f_3(t)\}$ and functions R_t^{ij} of time t which satisfy

$$
\begin{cases}
\dot{e}_1(t) = f_1(t), \\
\dot{e}_2(t) = e_1(t), \\
\dot{e}_3(t) = f_3(t), \\
\dot{f}_1(t) = -R_t^{11} e_1(t) - R_t^{31} e_3(t) - f_2(t), \\
\dot{f}_2(t) = -R_t^{22} e_2(t) - R_t^{32} e_3(t), \\
\dot{f}_3(t) = -R_t^{31} e_1(t) - R_t^{32} e_2(t) - R_t^{33} e_3(t).
\end{cases} \tag{6.1}
$$

Let δ_s be the dilation in the fibre direction defined by $\delta_s(\alpha) = s\alpha$ and let $\vec{E}$ be the Euler field defined by $\vec{E}(\alpha) = \frac{d}{ds}\Big|_{s=1} \delta_s \alpha$. By the definition of the Jacobi curve $J_\alpha(t)$, the time dependent vector field $(e^{t\vec{H}})^* \vec{E}$ is contained in $J_\alpha(t)$. Next we need the following lemma.

Lemma 6.1. $(e^{t\vec{H}})^* \vec{E} = \vec{E} - t\vec{H}$

Proof of Lemma 6.1. Using the definitions of the symplectic form ω and the Hamiltonian vector field $\vec{H}$, we have

$$\omega(d\delta_s(\vec{H}(\alpha)), X(s\alpha)) = s\omega(\vec{H}(\alpha), d\delta_{1/s}(X(s\alpha))) = -sdH(d\delta_{1/s}(X(s\alpha))).$$

Since the Hamiltonian is homogeneous of degree two in the fibre direction, the above equation becomes

$$\omega(d\delta_s(\vec{H}(\alpha)), X(s\alpha)) = -\frac{1}{s} dH(X(s\alpha)) = \frac{1}{s}\omega(\vec{H}(s\alpha), X(s\alpha)).$$

It follows that $\delta_s^* \vec{H} = s\vec{H}$, where $\delta_s^* \vec{H}$ is the pullback of the vector field $\vec{H}$ by the map δ_s. By comparing the flow of the above vector fields, we have

$$e^{t\vec{H}} \circ \delta_s = \delta_s \circ e^{ts\vec{H}}.$$

By differentiating the above equation with respect to s and set s to 1, it follows that $(e^{t\vec{H}})^* \vec{E} = \vec{E} - t\vec{H}$ as claimed. □

It follows from Lemma 6.1 that $\vec{E} - t\vec{H} = \sum_{i=1}^{3} a_i(t) e_i(t)$ for some functions a_i of time t. If we differentiate with respect to time t twice, we get

$$
\begin{aligned}
&2\dot{a}_1(t) f_1(t) + 2\dot{a}_2(t) e_1(t) + 2\dot{a}_3(t) f_3(t) - a_1(t)(R_t^{11} e_1(t) + R_t^{31} e_3(t) + \\
&+ f_2(t)) + a_2(t) f_1(t) - a_3(t)(R_t^{31} e_1(t) + R_t^{32} e_2(t) + R_t^{33} e_3(t)) + \\
&+ a_1(t) e_1(t) + a_2(t) e_2(t) + a_3(t) e_3(t) = 0.
\end{aligned}
$$

If we equate the coefficients of the f_i's, we get $a_1 \equiv a_2 \equiv \dot{a}_3 \equiv 0$. Therefore, $\vec{E} - t\vec{H} = a_3 e_3(t)$ and $-\vec{H} = a_3 f_3(t)$ for some constant a_3 satisfying $(a_3)^2 = \omega(a_3 f_3(t), a_3 e_3(t)) = dH(\vec{E}) = 2H$. It follows that $R^{31} = R^{32} = R^{33} = 0$.

7. Appendix 2: Optimal Transportation and the Generalized Curvature

In this appendix we discuss the relations between optimal transportation and the generalized curvature invariants. To do this, we first recall the displacement interpolation:

$$\varphi_t(x) = \pi(e^{t\vec{H}}(-df_x)),$$

where H is a Hamiltonian function on the cotangent bundle T^*M, $\vec{H}$ is the corresponding Hamiltonian vector field, $e^{t\vec{H}}$ is its Hamiltonian flow and f is a function which is twice differentiable at almost all points x. We also assume that the map φ_t is, at almost all points x in the manifold M, nonsingular for all t in $[0, 1)$.

We recall that the optimal maps to the optimal transportation problem (4.1) are of the form given by φ_1. Let μ be a smooth volume form on the manifold M and we denote the corresponding measure by the same symbol μ. Assume that the measures $\varphi_{t*}\mu$ are absolutely continuous with respect to the measure μ and let ρ_t be the corresponding density. In this appendix we describe the changes in the density ρ_t as a function of time t using the generalized curvature invariants. This is analogous to the Jacobi field calculations in [9, 26].

Recall that the vertical bundle V is given by the kernel of the map $d\pi : TT^*M \to TM$ and the Jacobi curve $J_\alpha(t)$ at α corresponding to a Hamiltonian H is defined by

$$J_\alpha(t) = de^{-t\vec{H}}(V_{e^{t\vec{H}}(\alpha)}).$$

Let $e_1(t), ..., e_n(t), f_1(t), ..., f_n(t)$ be a moving Darboux frame which satisfies

$$J_\alpha(t) = span\{e_1(t), ..., e_n(t)\}$$

and assume that the frame satisfies the following structural equations

$$\dot{e}_i(t) = \sum_j (\mathfrak{c}^1_{ij}(t)e_j(t)+\mathfrak{c}^2_{ij}(t)f_j(t)), \quad \dot{f}_i(t) = \sum_j (\mathfrak{c}^3_{ij}(t)e_j(t)+\mathfrak{c}^4_{ij}(t)f_j(t)). \tag{7.1}$$

Let $\mathfrak{C}^k_t$ be the matrix with entries equal to the structural constants $\mathfrak{c}^k_{ij}(t)$. Note that the moving Darboux frame $e_i(t)$, $f_i(t)$ and the structural constants $\mathfrak{c}^k_{ij}(t)$ depend on the point α in the manifold T^*M. Let $\mathfrak{m}$ be the n-form on the manifold T^*M which satisfies $i_{e_i(0)}\mathfrak{m} = 0$ and $\mathfrak{m}_\alpha(f_1(0), ..., f_n(0)) = 1$. A Hamiltonian H is *unimodular* with respect to a n-form η on the manifold M if there is a function $K : T^*M \to \mathbb{R}$ which is invariant under the Hamiltonian flow $e^{t\vec{H}}$ such that $\pi^*\eta = K\mathfrak{m}$.

We will also assume that the structural equations are *canonical*. To say precisely what it means, note that if $e_i(t)$ is a frame contained in the Jacobi curve $J_\alpha(t)$ at α, then $de^{s\vec{H}}(e_i(s+t))$ is a frame contained in the Jacobi curve $J_{e^{s\vec{H}}(\alpha)}(t)$ at $e^{s\vec{H}}(\alpha)$. Therefore, we can let $e_1(t), ..., e_n(t), f_1(t), ..., f_n(t)$ be a moving Darboux frame at α satisfying (7.1) and we define $\tilde{e}_i(t)$ and $\tilde{f}_i(t)$ by $\tilde{e}_i(t) := de^{s\vec{H}} e_i(s+t), \tilde{f}_i(t) := de^{s\vec{H}} f_i(s+t)$. The structural equations are canonical if $\tilde{e}_1(t), ..., \tilde{e}_n(t), \tilde{f}_1(t), ..., \tilde{f}_n(t)$ is a moving Darboux frame at $e^{s\vec{H}}(\alpha)$ satisfying

$$\dot{\tilde{e}}_i(t) = \mathfrak{c}^1_{ij}(t+s)\tilde{e}_j(t) + \mathfrak{c}^2_{ij}(t+s)\tilde{f}_j(t), \quad \dot{\tilde{f}}_i(t) = \mathfrak{c}^3_{ij}(t+s)\tilde{e}_j(t) + \mathfrak{c}^4_{ij}(t+s)\tilde{f}_j(t).$$

Let us denote the differential of the map $x \mapsto -df_x$ by $\mathcal{F}$, then the map $d\varphi_t$ satisfies $d\varphi_t = d\pi de^{t\vec{H}}\mathcal{F}$. If we let $\varsigma_i = d\pi(f_i(0))$, then the vectors $\mathcal{F}(\varsigma_1), ..., \mathcal{F}(\varsigma_n)$ span a linear subspace W of the symplectic vector space $T_\alpha T^*M$. We write $\mathcal{F}(\varsigma_i)$ as a linear combination with respect to the moving Darboux frame defined in (7.1):

$$\mathcal{F}(\varsigma_i) = \sum_{k=1}^{3} (a_{ij}(t)e_j(t) + b_{ij}(t)f_j(t)) \quad \text{or} \quad \Psi = A_t E_t + B_t F_t,$$

where $A_t = (a_{ij}(t))$, $B_t = (b_{ij}(t))$ and Ψ, E_t and F_t are matrices with rows $\mathcal{F}(\varsigma_i)$, $e_i(t)$ and $f_i(t)$ respectively.

Lemma 7.1. *Assume that the measures $\varphi_{t*}\mu$ is absolutely continuous with respect to μ, the Hamiltonian is unimodular with respect to μ and the structural equation is canonical, then the density ρ_t of $\varphi_{t*}\mu$ satisfies*

$$\rho_t(\varphi_t(x)) \det B_t = 1.$$

Proof. Assume that $e_1(t), ..., e_n(t), f_1(t), ..., f_n(t)$ is a moving Darboux frame at α which satisfies (7.1). Using the definition of $\tilde{e}_i$ and $\tilde{f}_i$, we have

$$de^{s\vec{H}}\mathcal{F}(\varsigma_i) = \sum_{k=1}^{n} (a_{ij}(s)\tilde{e}_j(0) + b_{ij}(s)\tilde{f}_j(0)).$$

Since the structural equations are canonical, it follows that

$$\mathfrak{m}(de^{s\vec{H}}\mathcal{F}(\varsigma_1), ..., de^{s\vec{H}}\mathcal{F}(\varsigma_n)) = \det B_s.$$

By the definition of the volume form η, the above expression implies that

$$\eta(d\varphi_s(\varsigma_1), ..., d\varphi_s(\varsigma_n)) = K(e^{s\vec{H}}\alpha) \det B_s = K(\alpha) \det B_s.$$

Since the function ρ_t is the density of the push forward measure $\varphi_{t*}\eta$ with respect to the measure μ (i.e. $\varphi_{t*}\eta = \rho_t\mu$), it follows that

$$K(\alpha)\det B_0 = \mu(\varsigma_1, ..., \varsigma_n) = K(\alpha)\rho_s(\varphi_s(x))\det B_s.$$

Since $\pi(-df_x) = x$, B_0 is the identity matrix and the proof is complete. $\square$

By assumption, for almost all points z in the manifold M, the map $d(\varphi_t)_z$ is nonsingular for all values of time t in $[0, 1)$. It follows that the density $\rho(\varphi_t(z))$ is nonzero for each such t. Lemma 7.1 shows that the corresponding matrix B_t is invertible and so the linear space W is transversal to the space $J_\alpha(t) = span\{e_1(t), ..., e_n(t)\}$. Therefore, W is the graph of a linear map from the space $span\{f_1(t), ..., f_n(t)\}$ to the space $J_\alpha(t) = span\{e_1(t), ..., e_n(t)\}$. Let S_t be the corresponding matrix. (i.e. the linear map is given by $f_i(t) \mapsto \sum_{i=1}^3 S_t^{ij} e_j(t)$, where S_t^{ij} are the entries of the matrix S_t.) Finally we come the main theorem of the appendix.

Theorem 7.2. *Suppose that the same assumptions as in Lemma 7.1 hold and assume further that, for almost all z in M, the map $d(\varphi_t)_z$ is nonsingular for all values of time t in $[0, 1)$. Then the matrix S_t satisfies the following matrix Riccati equation*

$$\dot{S}_t + \mathfrak{C}^3 + S_t\mathfrak{C}^1 - \mathfrak{C}^4 S_t - S_t\mathfrak{C}^2 S_t = 0$$

and the density ρ_t satisfies

$$\rho_t(\varphi_t(z)) = e^{\int_0^t tr(\mathfrak{C}^4 + S_s\mathfrak{C}^2)ds}.$$

Lemma 7.3. $S_t = B_t^{-1} A_t$.

Proof. Since $f_i(t) + \sum_{i=1}^3 S_t^{ij} e_j(t)$ is in the subspace W, $F_t + S_t E_t = P_t\Psi = P_t A_t E_t + P_t B_t F_t$ for some matrix P_t. By comparing the terms, we have $P_t A_t = S_t$ and $P_t B_t = I$. $\square$

Proof of Theorem 7.2. By differentiating $\Psi = B_t F_t + B_t S_t E_t$ with respect to time t, we get $B_t^{-1}\dot{B}_t F_t + \dot{F}_t + B_t^{-1}\dot{B}_t S_t E_t + \dot{S}_t E_t + S_t\dot{E}_t = 0$. If we apply the structural equations, then we get

$$\begin{cases} B_t^{-1}\dot{B}_t + \mathfrak{C}^4 + S_t\mathfrak{C}^2 = 0, \\ \dot{S}_t + B_t^{-1}\dot{B}_t S_t + \mathfrak{C}^3 + S_t\mathfrak{C}^1 = 0. \end{cases}$$

Therefore, S_t satisfies the equation

$$\dot{S}_t + \mathfrak{C}^3 + S_t\mathfrak{C}^1 - \mathfrak{C}^4 S_t - S_t\mathfrak{C}^2 S_t = 0.$$

Finally let $s_t = \rho_t(\varphi_t(x))$, then we have, by Lemma 7.1 and 7.3, the following:

$$\frac{1}{s_t}\frac{d}{dt}s_t = \det B_t \frac{d}{dt}\det(B_t^{-1}) = -tr(B_t^{-1}\dot{B}_t) = tr(\mathfrak{C}^4 + S_t\mathfrak{C}^2).$$

The rest of the theorem follows as claimed. □

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E-mail address: agrachev@sissa.it

International School for Advanced Studies via Beirut 4 - 34014, Trieste, Italy and Steklov Mathematical Institute, ul. Gubkina 8, Moscow, 119991 Russia

E-mail address: plee@math.toronto.edu

Department of Mathematics, University of Toronto, ON M5S 2E4, Canada